Unconstrained Extrema and Second Order Condition

Calculate the relative extrema of the following function: $z = e^{-x^2-2xy-y}$

Solution

We derive:

$$z'x = e^{-x^2-2xy-y}(-2x-2y) = 0$$

$$z'y = e^{-x^2-2xy-y}(-2x-1) = 0$$

Knowing that $e^{-x^2-2xy-y}$ can never be 0 then:

$$-2x-2y = 0$$

and

$$-2x-1 = 0$$

From this, we obtain that:

$$x = -y$$

and

$$x = -1/2$$

So we have: $(-1/2, 1/2, e^{-1/4+2/4-1/2}) = (1/2, 1/2, e^{-1/4})$

If we want to corroborate whether this point is a minimum or a maximum, we are going to calculate the Hessian matrix and analyze if this matrix is negative semidefinite or positive semidefinite.

$$H = \begin{pmatrix} z''xx & z''xy \\ z''yx & z''yy \end{pmatrix}$$

$$z'xx = (e^{-x^2-2xy-y}(-2x-2y))(-2x-2y) + e^{-x^2-2xy-y}(-2)$$

$$z'yy = (e^{-x^2-2xy-y}(-2x-1))(-2x-1) + e^{-x^2-2xy-y}(0)$$

$$z''xy = z''yx = e^{-x^2-2xy-y}(-2x-1)(-2x-2y) + e^{-x^2-2xy-y}(-2)$$

Evaluating each derivative at the point:

$$z''xx = -2e^{-1/4}$$

$$z''yy = 0$$

$$z''yx = z''xy = -2e^{-1/4}$$

Then the Hessian is: $H = \begin{pmatrix} -2e^{-1/4} & -2e^{-1/4} \\ -2e^{-1/4} & 0 \end{pmatrix}$ We have that $z'xx = -2e^{-1/4} < 0$ and the determinant of the Hessian is $|H| = -4e^{-1/2} < 0$. As the determinant of the Hessian is negative, we are dealing with a saddle point.